§6.3　定義に基づく定積分の計算

問題 6.3.1

$$\begin{aligned}
\int_1^4 x^2 dx &= (4-1)\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{4-1}{n}k\right)^2\right\} = 3\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{k=1}^{n}\left(\frac{n+3k}{n}\right)^2\right\} \\
&= 3\lim_{n\to\infty}\left\{\frac{1}{n}\frac{1}{n^2}\sum_{k=1}^{n}(n^2+6nk+9k^2)\right\} = 3\lim_{n\to\infty}\left\{\frac{1}{n^3}\left(\sum_{k=1}^{n}n^2+6n\sum_{k=1}^{n}k+9\sum_{k=1}^{n}k^2\right)\right\} \\
&= 3\lim_{n\to\infty}\left[\frac{1}{n^3}\left\{n^2 n+6n\frac{n}{2}(n+1)+9\frac{n}{6}(n+1)(2n+1)\right\}\right] \\
&= 3\lim_{n\to\infty}\left[\frac{1}{n^3}\left\{n^3+3n^2(n+1)+\frac{3n}{2}(n+1)(2n+1)\right\}\right] \\
&= 3\lim_{n\to\infty}\left\{1+3\left(1+\frac{1}{n}\right)+\frac{3}{2}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right\} = 3\left\{1+3(1+0)+\frac{3}{2}(2+0+0)\right\} \\
&= 21 \ .
\end{aligned}$$

問題 6.3.2　自然数 $k=1,2,3,\ldots,n$ について,

$$x_k - x_{k-1} = \xi_k = 2+\frac{2}{n}k-\left\{2+\frac{2}{n}(k-1)\right\} = \frac{2}{n} \ .$$

指数関数 3^x のリーマン和は,

$$\begin{aligned}
S_n &= \sum_{k=1}^{n}\left(3^{\xi_k}\frac{5}{n}\right) = \frac{2}{n}\sum_{k=1}^{n}3^{2+\frac{2}{n}(k-1)} = \frac{2}{n}\sum_{k=1}^{n}\left\{3^2 3^{\frac{2}{n}(k-1)}\right\} = \frac{18}{n}\sum_{k=1}^{n}3^{\frac{2}{n}(k-1)} \\
&= \frac{18}{n}\sum_{k=1}^{n}\left(3^{\frac{2}{n}}\right)^{k-1} = \frac{18}{n}\frac{\left(3^{\frac{2}{n}}\right)^n-1}{3^{\frac{2}{n}}-1} = \frac{18}{n}\frac{3^2-1}{3^{\frac{2}{n}}-1} = \frac{18}{n}\frac{8}{3^{\frac{2}{n}}-1} \\
&= \frac{144}{n}\frac{1}{3^{\frac{2}{n}}-1} \ .
\end{aligned}$$

ここで変数 x を $x=3^{\frac{2}{n}}-1$ とおく.　$3^{\frac{2}{n}}=1+x$,　$\frac{2}{n}=\log_3(1+x)$,　$\frac{144}{n}=72\log_3(1+x)$.

$$S_n = 72\log_3(1+x)\cdot\frac{1}{x} = 72\cdot\frac{1}{x}\log_3(1+x) = 72\log_3(1+x)^{\frac{1}{x}} \ .$$

$n\to\infty$ のとき $x=3^{\frac{2}{n}}-1\to 3^0-1=0$ なので,

$$\lim_{n\to\infty}S_n = \lim_{x\to 0}\left\{72\log_3(1+x)^{\frac{1}{x}}\right\} = 72\lim_{x\to 0}\log_3(1+x)^{\frac{1}{x}} \ ,$$

$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$ なので,

$$\lim_{x\to 0}\log_3(1+x)^{\frac{1}{x}} = \log_3\left\{\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right\} = \log_3 e \ .$$

従って,

$$\int_2^4 3^x dx = \lim_{n\to\infty}S_n = 72\log_3 e = \frac{72}{\ln 3} \ .$$