第7章の補遺1　指数関数と三角関数との積の不定積分

$\boxed{\text{問題 7.補遺1.1}}$　部分積分法により,

$$\int e^{2x}\sin 3x\,dx = \frac{1}{2}e^{2x}\sin 3x - \frac{1}{2}\int e^{2x}3\cos 3x\,dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx\ ,$$

$$\int e^{2x}\cos 3x\,dx = \frac{1}{2}e^{2x}\cos 3x - \frac{1}{2}\int e^{2x}(-3\sin 3x)\,dx = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx\ .$$

従って,

$$\begin{aligned}\int e^{2x}\sin 3x\,dx &= \frac{1}{2}e^{2x}\sin 3x - \frac{1}{2}\int e^{2x}3\cos 3x\,dx\\ &= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\left(\frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx\right)\\ &= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}\int e^{2x}\sin 3x\,dx\ .\end{aligned}$$

よって，積分定数を $A_1\,,B_1\,,C_1$ とおくと,

$$\int e^{2x}\sin 3x\,dx + \frac{9}{4}\int e^{2x}\sin 3x\,dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{3x}\cos 3x + A_1\ ,$$

$$\frac{13}{4}\int e^{2x}\sin 3x\,dx = e^{2x}\left(\frac{1}{2}\sin 3x - \frac{3}{4}\cos 3x\right) + B_1\ ,$$

故に

$$\int e^{2x}\sin 3x\,dx = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C_1 \quad (\ C_1\ \text{は積分定数})\ .$$

また,

$$\begin{aligned}\int e^{2x}\cos 3x\,dx &= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx\\ &= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left(\frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx\right)\\ &= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}\int e^{2x}\cos 3x\,dx\ .\end{aligned}$$

よって，積分定数を $A_2\,,B_2\,,C_2$ とおくと,

$$\int e^{2x}\cos 3x\,dx + \frac{9}{4}\int e^{2x}\cos 3x\,dx = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x + A_2\ ,$$

$$\frac{13}{4}\int e^{2x}\cos 3x\,dx = e^{2x}\left(\frac{3}{4}\sin 3x + \frac{1}{2}\cos 3x\right) + B_2\ ,$$

故に

$$\int e^{2x}\cos 3x\,dx = \frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C_2 \quad (\ C_2\ \text{は積分定数})\ .$$