§6.8　余弦定理

問題 6.8.1　余弦定理より

$$\overline{\mathrm{AB}}^2 = 4^2 + \sqrt{3}^2 - 2\cdot 4\cdot\sqrt{3}\cdot\cos 30^\circ = 16 + 3 - 8\sqrt{3}\cdot\frac{\sqrt{3}}{2} = 7\ ,$$

$\overline{\mathrm{AB}} \geq 0$ なので $\overline{\mathrm{AB}} = \sqrt{7}$.

問題 6.8.2

$$\cos 150^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}\ .$$

余弦定理より

$$\overline{\mathrm{AB}}^2 = 5^2 + \left(2\sqrt{3}\right)^2 - 2\cdot 5\cdot 2\sqrt{3}\cdot\cos 150^\circ = 25 + 12 - 20\sqrt{3}\cdot\left(-\frac{\sqrt{3}}{2}\right) = 67\ ,$$

$\overline{\mathrm{AB}} \geq 0$ なので $\overline{\mathrm{AB}} = \sqrt{67}$.

問題 6.8.3　$q = \overline{\mathrm{AC}}$ とおく.　余弦定理より $13 = q^2 + 25 - 2\cdot 5\cdot q\cdot\cos 45^\circ$, $q^2 - 5\sqrt{2}\,q + 12 = 0$, $(q - 2\sqrt{2})(q - 3\sqrt{2}) = 0$, $q = 2\sqrt{2}$ または $b = 3\sqrt{2}$.　つまり $\overline{\mathrm{AC}} = 2\sqrt{2}$ または $\overline{\mathrm{AC}} = 3\sqrt{2}$.

問題 6.8.4

$$\cos 120^\circ = \cos(30^\circ + 90^\circ) = -\sin 30^\circ = -\frac{1}{2}\ .$$

余弦定理より

$$\begin{aligned}\overline{\mathrm{AB}}^2 &= \overline{\mathrm{AC}}^2 + \overline{\mathrm{BC}}^2 - 2\overline{\mathrm{AC}}\,\overline{\mathrm{BC}}\cos\angle\mathrm{C} = 2^2 + \left(\sqrt{3} - 1\right)^2 - 2\cdot 2\cdot\left(\sqrt{3} - 1\right)\cos 120^\circ \\ &= 4 + 3 - 2\sqrt{3} + 1 - 4\left(\sqrt{3} - 1\right)\left(-\frac{1}{2}\right) = 8 - 2\sqrt{3} + 2\sqrt{3} - 2 \\ &= 6\ ,\end{aligned}$$

よって $\overline{\mathrm{AB}} = \sqrt{6}$.　余弦定理より $\overline{\mathrm{AC}}^2 = \overline{\mathrm{AB}}^2 + \overline{\mathrm{BC}}^2 - 2\overline{\mathrm{AB}}\,\overline{\mathrm{BC}}\cos\angle\mathrm{B}$ なので,

$$2^2 = \sqrt{6}^{\,2} + \left(\sqrt{3} - 1\right)^2 - 2\sqrt{6}\left(\sqrt{3} - 1\right)\cos\angle\mathrm{B}\ ,$$

$$4 = 6 + 3 - 2\sqrt{3} + 1 - 2\sqrt{6}\left(\sqrt{3} - 1\right)\cos\angle\mathrm{B}\ ,$$

$$2\sqrt{6}\left(\sqrt{3} - 1\right)\cos\angle\mathrm{B} = 2(3 - \sqrt{3})\ ,$$

$$\cos\angle\mathrm{B} = \frac{3 - \sqrt{3}}{\sqrt{6}\left(\sqrt{3} - 1\right)} = \frac{\sqrt{3}\left(\sqrt{3} - 1\right)}{\sqrt{6}\left(\sqrt{3} - 1\right)} = \frac{\sqrt{3}}{\sqrt{6}} = \frac{1}{\sqrt{2}}\ ,$$

よって $\angle\mathrm{B} = 45^\circ$.

問題 6.8.5

(1)　余弦定理より

$$\cos\angle\mathrm{A} = \frac{4^2 + 7^2 - 9^2}{2\cdot 4\cdot 7} = \frac{-16}{2\cdot 4\cdot 7} = -\frac{2}{7}\ .$$

(2)　$(\sin\angle\mathrm{A})^2 + (\cos\angle\mathrm{A}) = 1$ なので,

$$\left(\sin\angle\mathrm{A}\right)^2 = 1 - (\cos\angle\mathrm{A}) = 1 - \left(\frac{2}{7}\right)^2 = \frac{45}{49}\ ,$$

$0^\circ \leq \angle\mathrm{A} \leq 180^\circ$ より $\sin\angle\mathrm{A} \geq 0$ なので, $\sin\angle\mathrm{A} = \sqrt{\dfrac{45}{49}} = \dfrac{\sqrt{45}}{7}$.

(3)　三角形 ABC の面積は

$$\frac{1}{2}\overline{\mathrm{AB}}\,\overline{\mathrm{AC}}\sin\angle\mathrm{A} = \frac{1}{2}\cdot 4\cdot 7\cdot\frac{\sqrt{45}}{7} = 2\cdot 3\sqrt{5} = 6\sqrt{5}\ .$$